ACADIMIC
World Academic Union

ISSN 1749-3889(print),1749-3897(online)
International Journal of Nonlinear Science
Vol.3 (2007) No.2, pp. 91-102

Stability Criteria of Difference Equations

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(Received 3 January 2007, accepted 16 January 2007)

Abstract: In this paper,we will discuss eventual stability of difference equations systems and extend it to the so-called eventual ϕ_0- stability of difference equations systems.given some criteria and results. Our technique depends on Liapunov's direct method,cone-valued Liapunov's function method and comparison technique. *(MSC)-34D20.*

Keywords: Eventual stability; ϕ_0 -stability ;Eventual ϕ_0 -stability ;Uniform eventual ϕ_0 -stability; Uniform asymptotic eventual ϕ_0-stability: Liapunov function.

1 Introduction

The qualitative theory has been considerable interest in studying and improving emphasis stability of difference equations systems(see[2, 3, 8–12].Lakshmikantham and Leela[7] initiated the method of cone and cone- valued Liapunov function.

Recently,ϕ_0-stability has been discussed in comparison differential systems, and some criteria were given(see[1]). We can refer to the paper of El-Shiekh et al[4, 5],Soliman[9], and Wang et al[10].

Our purpose in this paper is to discuss eventual stability and extend it to eventual ϕ_0- stability for difference equations systems which lie somewhere between eventual stability on one side and ϕ_0- stability of [10]on the othere side via cone-valued Liapunov function method that was studied in[1]and used in[10].

The paper is organized as follows.In section one,we introduce some preliminary, definitions and notations which will be used in this paper.In section two,we discuss the notion of eventual stability of difference systems using Liaponuv's second method.In Section three we extend eventual stability to eventual ϕ_0- stability of difference systems using cone- valued Liapunov function method.In section four,we discuss eventual ϕ_0- stability using comparison technique with the comparison difference system.

Let $\Re^m$ be the m-dimensional Euclidean real space,$J = [n_0, \infty)$, and $\Re^+ = [0, \infty)$.

Consider systems of difference equations

$$x(n + 1) = f(n, x(n)), \;\; n \in N^+, \tag{1.1}$$

where $f \in C[N^+ \times \Re^m, \Re^m]$ is continuous functions in $x(n), x(n) \in \Re^m$, and Z is the set of integers $f(n, 0) = 0$ for $n \in Z$,and

$$N^+ = \{n : n \in Z, n \geq 0\},$$

so that the system (1.1) always has the zero solution $x(n) = 0$. Thus for any given $n_0 \in N^+$,there is a unique solution of (1.1) denoted by $x(n) = x(n, n_0, x_0)$ such that it satisfied (1.1) for all integer $n \geq n_0$ and

$$x(n_0, n_0, x_0) = x_0, \quad \text{for} \quad n \geq n_0.$$

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IJNS.2007.04.15/066

The contiuity of f implies the continuity of any solutions of (1.1).

The following definitions will be needed.

Definition 1.1[3]. A function $b(r)$ belongs to the class $\mathcal{K}$ if $b(r) \in C[(0,\rho),\Re^+], b(0) = 0$ and $b(r)$ is strictly monotone increasing in r.

Definition 1.2[1]. A proper subset $K \subset \Re^m$ is called a cone if $(i)\lambda K \subset K, \lambda \geq 0, \quad (ii)K + K \subset K, \quad (iii)\overline{K} = K, \quad (iv)K^\circ \neq \emptyset, \quad (v)K \cap (-K) = 0$, where $\overline{K}$ and K° denote the closure and interior of K respectively, and ∂K denotes the boundary of $K, x \in \partial K \Longleftrightarrow y - x = 0 \quad$ for some $\quad y \in K_0^*, K_0 = K - 0$.

The order relation on $\Re^m$ induced by the cone K is defined as follows.

Let $x, y \in K$,then

$$x \leq_K y \Longleftrightarrow y - x \in K, \text{and} \quad x \leq_{K^o} y \Longleftrightarrow y - x \in K^o.$$

The set K^* is called the adjoint cone if $K^* = \{\phi \in \Re^m : \quad (\phi, x) \geq 0\}$, for $x \in K$, satisfies the properties $(i) - (v)$ of definition 1.2.

Definition 1.3[1].A function $g : D \to \Re^m, D \subset \Re^m$ is called quasimonotone relative to the cone K, If $\quad x, y \in D \quad$ and $\quad y - x \in \partial K$, then there exists $\phi_\circ \in K_\circ^\star$ such that $(\phi_\circ, y - x) = 0$ and $(\phi_\circ, g(y) - g(x)) \geq 0$.

Let K be a cone of $\Re^m$. Define

$$S_\rho = \{x \in \Re^m :\parallel x \parallel \leq \rho, \quad \rho > 0\},$$

and $V(n, x) \in C[N^+ \times S_\rho, K]$; we define $\triangle V(n, x)$ as follows :

$$\begin{aligned}\triangle V(n, x)_{(1.1)} &= V(n + 1, x(n + 1)) - V(n, x(n)) \\ &= V(n + 1, f(n, x)) - V(n, x(n))\end{aligned}$$

Consider the comparison difference system

$$u(n + 1) = G(n, u(n)), \quad u(n_0) = u_0, \tag{1.2}$$

where $G \in [N \times K, \Re^m], u(n) = u(n, n_0, u_0)$ is a solution of (1.2) for (n_0, u_0). For $V(n, u) \in C[N^+ \times S_\rho, K]$; the definition of the difference operator $\triangle V(n, u)$ is defined as follows :

$$\begin{aligned}\triangle V(n, u) \mid_{(1.2)} &= V(n + 1, u(n + 1)) - V(n, u(n)) \\ &= V(n + 1, G(n, u)) - V(n, u(n))\end{aligned}$$

As in [10], the scalar product of a, b is define as

$$(a, b) = \sum_{i=1}^{n} a_i b_i \quad \text{for} \quad a, b \in \Re^m.$$

Definition 1.4 [10]. The zero solution of (1.1) is said to be ϕ_0-stable if for each $\epsilon > 0, n_0 \in N^+$, there exists positive function $\delta(n_0, \epsilon) > 0$ that is continuous in n_0, such that for $\phi_0 \in K_0^*$

$$(\phi_0, x^*(n, n_0, x_0)) < \epsilon, \quad \text{for} \quad n \geq n_0.$$

provided $(\phi_0, x_0) \leq \delta$.

Where x^* here and in this paper denote the maximal solution of (1.1) relative to the cone $K \subset \Re^m$.

The following definitions are some what new and related with that of [10].

Definition 1.5. The zero solution of (1.1)is said to be eventually stable if for each $\epsilon > 0, n_0 \in N^+$, there exists positive function $\tilde{n}(n_0, \epsilon) > 0$, and $\delta(n_0, \epsilon) > 0$ that is continuous in n_0, such that

$$\| x(n, n_0, x_0) \| < \epsilon, \quad \text{for} \quad n \geq n_0 \geq \tilde{n}.$$

provided that $\| \psi \| \leq \delta$.

Definition 1.6. The zero solution of (1.1)is said to be eventually ϕ_0-stable if for each $\epsilon > 0, n_0 \in N^+$, there exist positive functions $\tilde{n}(n_0, \epsilon) > 0$, and $\delta(n_0, \epsilon) > 0$ that is continuous in n_0, such that for $\phi_0 \in K_0^*$

$$(\phi_0, x^*(n, n_0, x_0)) < \epsilon, \quad \text{for} \quad n \geq n_0.$$

provided that $(\phi_0, x_0) < \delta$.

Definition 1.7. The zero solution of (1.2) is said to be eventually asymptotically ϕ_0-stable if it is ϕ_0-stable and for each $\epsilon > 0, n_0 \in N^+$, and $T = T(n_0, \epsilon)$ such that

$$(\phi_0, x_0) < \delta \Rightarrow (\phi_0, x^*(n, n_0, x_0)) < \epsilon, \quad \text{for} \quad n \geq \tilde{n} + T(n_0, \epsilon).$$

If it is uniformly eventually asymptotically ϕ_0-stable, δ and T are independent of n_0.

2 Eventual stability

In this section,we will discuss and obtain some results of eventual stability of the difference system (1.1)using Liapunov direct method.

Theorem 2.1.Assume that there exist functions $V(n, x) \in C[N^+ \times S_\rho, K]$, and $G \in [N^+ \times S_\rho, \Re^m]$ such that $V(n, 0) = 0, G(n, 0) = 0$ satisfying the conditions

$A_1)$ $a \| x(n, n_0, x_0) \| \leq V(n, x), \quad a \in \mathcal{K}$,

$A_2)$ $\triangle V(n, x) \leq 0$,

$A_3)$ $f(n, x)$ is Lipschitzian in x.

Then the zero solution of (1.1) is eventually stable

Proof.From the continuity of $V(n, x)$ and $V(n, 0) = 0$, thus given $a_1(\epsilon) > 0, \delta_0 \in N^+$, there exists $\delta(n_0, \epsilon) > 0$ such that

$$\| x_0 \| < \delta_1 \Rightarrow \| V(n_0, x_0) \| < a_1(\epsilon) \tag{2.1}$$

From the condition (A_2),we get

$$\begin{aligned}
&\| V(n, x)) - V(n_0, x_0) \| \\
&= \| V(n, x) - V(n-1, x(n-1)) + V(n-1, x(n-1)) \\
&\qquad - V(n-2, x(n-2)) + V(n-2, x(n-2)) + ... \\
&\qquad + V(n_0+1, x(n_0+1)) - V(n_0, x_0) \| \\
&= \triangle V(n-1, x(n-1)) + \triangle V(n-2, x(n-2)) + ... \\
&\qquad + \triangle V(n_0+1, x(n_0+1)) - \triangle V(n_0, x_0) \\
&= \sum_{j=n_0}^{j=n-1} \triangle V(j, x(j)) \leq 0.
\end{aligned}$$

Then,we have

$$V(n, x) \leq V(n_0, x_0). \tag{2.2}$$

From the condition (A_1), and (2.2) we have

$$a \| x(n) \| \leq V(n_0, x_0) \leq a(\epsilon), \quad a \in \mathcal{K}, \tag{2.3}$$

provided that $\| x_0 \| \leq \delta, \quad \text{for} \quad n \geq n_0 \geq \tilde{n}, \quad \tilde{n} > 0$. i.e,

$$\| x_0 \| \leq \delta(n_0, \epsilon) \Rightarrow \| x(n) \| < \epsilon, \quad \text{for} \quad n \geq n_0 \geq \tilde{n}, \quad \tilde{n} > 0. \tag{2.4}$$

Then the zero solution of (1.1)is eventually stable.

Theorem 2.2.Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K]$, and $G \in [N^+ \times K, \Re^m]$ such that $V(n,0)=0, G(n,0)=0$ satisfying the conditions $(A_1),(A_2),(A_3)$ of Theorem 2.1 and

$A_4)\quad \triangle V(n,x) \leq b(\| x^*(n,n_0,x_0) \|), \quad b \in \mathcal{K}$

Then the zero solution of (1.1)is uniformly eventually stable.

Proof.From Theorem 2.1,the zero solution of (1.1) is eventually stable.The conditions (A_1),and (A_4) yield

$$a(\| x(n) \|) \leq V(n,x) \leq V(n_0,x_0) \leq b(\| x_0 \|), \quad a,b \in \mathcal{K}$$

Choosing $\delta = b^{-1}(a(\epsilon))$ indepenrent of n_0,we have

$$\begin{aligned} a(\| x(n) \|) \leq V(n,x) &\leq V(n_0,x_0) \\ &< b[(\phi_0,,x_0)] \leq b(\delta) = b(b^{-1}(a(\epsilon))) = a(\epsilon). \end{aligned}$$

Thus

$$\| x_0 \| \leq \delta(\epsilon) \Rightarrow \| x(n) \| < \epsilon, \quad n \geq n_0 \geq \tilde{n}. \tag{2.5}$$

Then the zero solution of (1.1)is uniformly eventually stable.

Theorem 2.3.Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K], and\ G \in [N^+ \times K, \Re^m]$ such that $V(n,0)=0, G(n,0)=0$ satisfying the conditions $(A_1),(A_3),(A_4)$ and

$A_5)\quad \triangle V(n,x) \leq -c[V(n,x)], \quad c \in \mathcal{K},$

Then the zero solution of (1.1)is eventually asymptotically stable.

Proof.The condition(A_5)implies the condition (A_2) of Theorem 2.1,thus by Theorem 2.1,the zero solution of (1.1)is eventually stable.

From the condition(A_5),we get

$$\begin{aligned} &\| V(n+1,x(n+1)) - V(n_0,x_0) \| \\ =&\| V(n+1,x(n+1)) + V(n,x(n)) - V(n,x(n)) \\ &\quad + V(n-1,x(n-1)) - V(n-1,x(n-1)) + ... \\ &\quad + V(n_0+1,x(n_0+1)) - V(n_0,x_0) \| \\ =& \sum_{j=n_0}^{j=n} \triangle V(j,x(j)) \\ \leq& -\sum_{j=n_0}^{j=n} c[V(j,x(j))]. \end{aligned}$$

Thus

$$V(n+1,x(n+1)) + \sum_{j=n_0}^{j=n} c[V(j,x(j))] \leq V(n_0,x_0) \tag{2.6}$$

From the condition (A_1),and (2.6) as $n \to \infty$,we have

$$\sum_{j=n_0+1}^{j=\infty} c(\| x(j) \|) \leq V(n_0,x_0).$$

Goingthrough in [10],and using the property of the convergence of the progression yield

$$c[x(j)] \to 0, \quad \text{as} \quad j \to \infty, \quad c \in \mathcal{K},$$

which implies that $\| x(j) \| \to 0, \quad$ as $\quad j \to \infty$.

Then the zero solution of (1.1)is eventually asymptotically stable.

Theorem 2.4.Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K]$,and $G \in [N^+ \times K, \Re^m]$ such that $V(n,0)=0, G(n,0)=0$ satisfying the conditions $(A_1),(A_3),(A_4)$ of Theorem 2.1 and

$A_6)$ $(\phi_0, \triangle V(n,x)) \leq -c[(\phi_0, x^*(n,n_0,x_0))], \quad c \in \mathcal{K}$,

Then the zero solution of (1.1)is uniformly eventually asymptotically stable

Proof.The condition(A_2) implies the condition(A_2),therefore the conditions of Theorem 2.1 are all satisfied,and the zero solution of (1.1) is eventually stable,i,e. for each $\epsilon > 0$,there exist $\delta(n_0,\epsilon), \tilde{n}(n_0,\epsilon)$ such that

$$\| x_0 \| < \delta(\epsilon) \Rightarrow \| x \| < \epsilon, \quad n \geq n_0 \geq \tilde{n}$$

To prove rest of the proof,it must be show that for each $\epsilon > 0$, there exist $\delta(\epsilon), \alpha$ and $T(\epsilon) \in N^+$ such that

$$\| x \| < \epsilon, \quad n \geq n_0 + T(\epsilon), \tilde{n} \geq n_0, \quad \text{for} \quad \| x_0 \| \leq \alpha.$$

Now, we choose

$$T = \hat{E} + n_0 \tag{2.7}$$

where $E = \frac{b(\delta)}{c(\delta)}$ and $\hat{E}$ denotes the maximal integer of the number E.For $(\phi_0, \psi)) < \alpha$,then there must exist a $n_1 \in [n_0, n_0+T]$ such that

$$\| x(n,n_0,x_0) \| < \delta.$$

Suppose that this is not true,thus for $n \in [n_0, n_0+T]$,we have

$$\| x(n,n_0,x_0) \| \geq \delta,$$

and thus

$$(\phi_0, \triangle V(n,x)) \leq -c[(\phi_0, x^*(n))] \leq -c(\delta),$$

therefore

$$\begin{aligned} \| V(T+1, x(T+1))) - V(n_0,x_0) \| &= \sum_{j=n_0}^{j=T} c[V(j,x(j))] \\ &\leq -c(\delta)(T-n_0), \end{aligned}$$

and from(2.7), we get

$$\begin{aligned} V(T+1, x(T+1)) &\leq V(n_0,x_0) - c(\delta)(T-n_0) \\ &\leq b(\| x_0 \|) - c(\delta)(T-n_0) \\ &\leq b(\delta) - c(\delta)\hat{E} < 0, \end{aligned}$$

This contradicts the condition (A_1). Thus there exists $n_1 \in [n_0, n_0+T]$ such that

$$\| x(n_1,n_0,x_0) \| < \delta,$$

and from the uniform eventual stability of the zero solution of (1.1), we have

$$\| x_0) \| \leq \delta(\epsilon) \Rightarrow \| x(n,n_0,x_0) \| < \epsilon, \quad n \geq n_0 + T(\epsilon).$$

Then the zero solution of (1.1)is uniformly eventually asymptotically stable.

3 Eventual ϕ_0 -stability

In this section,we will discuss and obtain some results of eventual ϕ_0- stability of the difference system (1.1)using cone-valued Liapunov function method of[7].

Theorem 3.1.Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K], and\ G \in [N^+ \times K, \Re^m]$ such that $V(n,0)=0, G(n,0)=0$ satisfying the conditions

$H_1)$ $a[(\phi_0, x^*(n,n_0,x_0))] \leq (\phi_0, V(n,x)), \quad a \in \mathcal{K}$,

H_2) $(\phi_0, \triangle V(n,x)) \leq 0$,
H_3) $f(n,x)$ is quasimonotone in x relative to K.
Then the zero solution of (1.1) is eventually ϕ_0 -stable

Proof.From the continuity of $V(n,x)$ and $V(n,0) = 0$,and given $a_1(\epsilon) > 0, \delta_0 \in N^+$,there exists $\delta(n_0,\epsilon) > 0$ such that

$$\| x_0 \| < \delta_1 \Rightarrow \| V(n_0,x_0) \| < a_1(\epsilon). \tag{3.1}$$

For $\phi_0 \in K_0^*$,thus

$$\| \phi_0 \| \| x_0 \| < \| \phi_0 \| \delta_1 \Rightarrow \| \phi_0 \| \| V(n_0,x_0) \| < \| \phi_0 \| a_1(\epsilon). \tag{3.2}$$

Now,for

$$(\phi_0, x_0) \leq \| \phi_0 \| \| x_0 \|, \quad (\phi_0, V(n_0,x_0)) \leq \| \phi_0 \| \| V(n_0,x_0) \|$$

and $x_0, V(n_0,x_0) \in K$,we have

$$(\phi_0, x_0) \geq 0 \quad \text{and} \quad (\phi_0, V(n_0,x_0)) \geq 0,$$

it follows that

$$(\phi_0, x_0) \leq \delta \quad \Rightarrow (\phi_0, V(n_0,x_0)) \leq a(\epsilon), \tag{3.3}$$

where $\delta = \| \phi_0 \| \delta_1, \quad a(\epsilon) = \| \phi_0 \| a_1(\epsilon)$.From the condition($H_2$),we get

$$\begin{aligned}
&(\phi_0, V(n,x)) - (\phi_0, V(n_0,x_0)) \\
&= (\phi_0, V(n,x)) - (\phi_0, V(n-1, x(n-1))) + (\phi_0, V(n-1, x(n-1))) \\
&\qquad - (\phi_0, V(n-2, x(n-2))) + (\phi_0, V(n-2, x(n-2))) + ... \\
&\qquad + (\phi_0, V(n_0+1, x(n_0+1))) - (\phi_0, V(n_0,x_0)) \\
&= (\phi_0, \triangle V(n-1, x(n-1))) + (\phi_0, \triangle V(n-2, x(n-2))) + ... \\
&\qquad + (\phi_0, \triangle V(n_0+1, x(n_0+1))) - (\phi_0, \triangle V(n_0,x_0)) \\
&= \sum_{j=n_0}^{j=n-1} (\phi_0, \triangle V(j, x(j))) \leq 0.
\end{aligned}$$

Then,we have

$$(\phi_0, V(n,x)) \leq (\phi_0, V(n_0,x_0)). \tag{3.4}$$

From the condition (H_1), (3.3), and (3.4) we have

$$a[(\phi_0, x(n))] \leq (\phi_0, V(n_0,x_0)) \leq a(\epsilon), \quad a \in \mathcal{K}, \tag{3.5}$$

provided that $(\phi_0, x_0) < \delta, \quad$ for $\quad n \geq n_0 \geq \tilde{n}, \quad \tilde{n} > 0$. i.e,

$$(\phi_0, x_0) < \delta(n_0,\epsilon) \Rightarrow (\phi_0, x(n)) < \epsilon, \quad \text{for} \quad n \geq n_0 \geq \tilde{n}, \quad \tilde{n} > 0. \tag{3.6}$$

Then the zero solution of (1.1) is eventually ϕ_0-stable.

Theorem 3.2.Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K]$, and $G \in [N^+ \times K, \Re^m]$ such that $V(n,0) = 0, G(n,0) = 0$ satisfying the conditions $(H_1, (H_2, (H_3)$ of Theorem 3.1 and
H_4) $(\phi_0, \triangle V(n,x)) \leq b[(\phi_0, x^*(n,n_0,x_0))], \quad b \in \mathcal{K}$.
Then the zero solution of (1.1)is uniformly eventually ϕ_0 -stable.

Proof. From Theorem 2.1, the zero solution of (1.1) is eventually ϕ_0 -stable. The conditions (H_1), and (H_4) yield

$$a[(\phi_0, x(n))] \leq (\phi_0, V(n,x)) \leq (\phi_0, V(n_0,x_0)) \leq b[(\phi_0, x_0)], \quad a, b \in \mathcal{K}$$

Choosing $\delta = b^{-1}(a(\epsilon))$ indepenrent of n_0, we have

$$\begin{aligned} a[(\phi_0, x(n))] &\leq (\phi_0, V(n,x)) \\ &\leq (\phi_0, V(n_0, x_0)) \\ &< b[(\phi_0, , x_0)] \\ &\leq b(\delta) \\ &= b(b^{-1}(a(\epsilon))) = a(\epsilon). \end{aligned}$$

Thus

$$(\phi_0, x_0) \leq \delta(\epsilon) \Rightarrow (\phi_0, x(n)) < \epsilon, n \geq n_0 \geq \tilde{n}, \tilde{n} > 0. \tag{3.7}$$

Then the zero solution of (1.1)is uniformly eventually ϕ_0 -stable.

Theorem 3.3. Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K]$, and $G \in [N^+ \times K, \Re^m]$ such that $V(n,0) = 0, G(n,0) = 0$ satisfying the conditions $(H_1, (H_3), (H_4)$ and
$H_5)$ $(\phi_0, \triangle V(n,x)) \leq -c[(\phi_0, V(n,x))], \quad c \in \mathcal{K},$
Then the zero solution of (1.1)is eventually asymptotically ϕ_0 equistable.

Proof. The condition (H_6) implies the condition (H_2) of Theorem 3.1, thus by Theorem 3.1, the zero solution of (1.1) is ϕ_0 -equistable. From the condition (H_5), we get

$$\begin{aligned} &(\phi_0, V(n+1, x(n+1))) - (\phi_0, V(n_0, x_0)) \\ &= (\phi_0, V(n+1, x(n+1))) + (\phi_0, V(n, x(n))) - (\phi_0, V(n, x(n))) \\ &\qquad + (\phi_0, V(n-1, x(n-1))) - (\phi_0, V(n-1, x(n-1))) + ... \\ &\qquad + (\phi_0, V(n_0+1, x(n_0+1))) - (\phi_0, V(n_0, x_0)) \\ &= \sum_{j=n_0}^{j=n} (\phi_0, \triangle V(j, x(j))) \\ &\leq - \sum_{j=n_0}^{j=n} c[(\phi_0, V(j, x(j))). \end{aligned}$$

Thus

$$(\phi_0, V(n+1, x(n+1))) + \sum_{j=n_0}^{j=n} c[(\phi_0, V(j, x(j))) \leq (\phi_0, V(n_0, x_0)) \tag{3.8}$$

From the condition (H_1), and (3.8) as $j \to \infty$, we have

$$\sum_{j=n_0+1}^{j=\infty} c[(\phi_0, x(j)) \leq (\phi_0, V(n_0, x_0)).$$

Samilarly as goingthrough in Theorem 2.3,we have

$$c[(\phi_0, x(j)) \to 0 \Rightarrow (\phi_0, x(j)) \to 0, \quad \text{as} \quad j \to \infty, c \in \mathcal{K}, \phi_\prime \in \mathcal{K}_\prime^*.$$

Then the zero solution of (1.1)is eventually asymptotically ϕ_0 -stable.

Theorem 3.4. Assume that there exist functions $V(n,x) \in C[N^+ \times S_\rho, K]$, and $G \in [N^+ \times K, \Re^m]$ such that $V(n,0) = 0, G(n,0) = 0$ satisfying the conditions $(H_1, (H_3), (H_4)$ of Theorem 3.1 and
$H_6)$ $(\phi_0, \triangle V(n,x)) \leq -c[(\phi_0, x^*(n, n_0, x_0))], \quad c \in \mathcal{K},$
Then the zero solution of (1.1)is uniformly eventually asymptotically ϕ_0 -stable.

Proof.The condition (H_6) implies the condition(H_2), therefore the conditions of
Theorem 3.1 are all satisfied, then the zero solution of (1.1) is eventually ϕ_0- stable, i,e. for each $\epsilon > 0$, there exist $\delta(\epsilon)$ such that

$$(\phi_0, x_0) < \delta(\epsilon) \Rightarrow (\phi_0, x^*(n, n_0, x_0)) < \epsilon, \quad n \geq n_0 \geq \tilde{n}.$$

To complete the proof, it must be shown that for each $\epsilon > 0$, there exist $\delta(\epsilon), \alpha$ and $T(\epsilon) \in N^+$ such that

$$(\phi_0, x^*(n, n_0, x_0)) < \epsilon, \quad n \geq n_0 + T(\epsilon), \tilde{n} \geq n_0, \quad \text{for} \quad (\phi_0, x_0)) < \alpha. \tag{3.9}$$

For $(\phi_0, x_0)) < \alpha$, then there must exist an $n_1 \in [n_0, n_0 + T]$ such that

$$(\phi_0, x^*(n_1, n_0, x_0)) < \delta.$$

Suppose that this is not true,thus for $n \in [n_0, n_0 + T]$, we have

$$(\phi_0, x^*(n, n_0, x_0)) \geq \delta,$$

and thus

$$(\phi_0, \triangle V(n, x)) \leq -c[(\phi_0, x^*(n))] \leq -c(\delta),$$

Therefore by (2.7), we get

$$\begin{aligned}(\phi_0, V(T+1, x(T+1))) - V(n_0, x_0) &= \sum_{j=n_0}^{j=T} c[(\phi_0, V(j, x(j))) \\ &\leq -c(\delta)(T - n_0),\end{aligned}$$

i,e.

$$\begin{aligned}(\phi_0, V(T+1, x(T+1))) &\leq V(n_0, x_0) - c(\delta)(T - n_0) \\ &\leq b[(\phi_0, x_0)] - c(\delta)(T - n_0) \\ &\leq b(\delta) - c(\delta)\overline{\left[\frac{b(\delta)}{c(\delta)}\right]} < 0,\end{aligned}$$

This contradicts the condition (H_1). Thus there exists $n_1 \in [n_0, n_0 + T]$ such that

$$(\phi_0, x^*(n_1, n_0, x_0)) < \delta,$$

and from the uniform eventual ϕ_0-stability of the zero solution of (1.1), we have

$$(\phi_0, x_0)) < \delta(\epsilon) \Rightarrow (\phi_0, x^*(n, n_0, x_0)) < \epsilon.$$

Then the zero solution of (1.1) is uniformly eventually asymptotically ϕ_0-stable.

4 Comparison Theorems

In this section,we will discuss and obtain some results of eventual ϕ_0- stability of the difference system (1.1) and will compare it with difference system (1.2).

The following result without proof will be used in the sequel

Lemma 4.1[10] If there exists a function $V(n, x) \in C[N^+ \times S_\rho, K]$ such that the function along the solution of (1.1) satisfies

$$(\phi_0, V(T+1, x(T+1))) \leq (\phi_0, G(n, V(n, x(n)))), \quad \text{for some} \quad \phi_0 \in K_0^* \tag{4.1}$$

where $G \in C[N^+ \times K, \Re^m]$, and $G(n, u)$ is quasimonotone increasing in u relative to K, then

$$(\phi_0, V(n_0, x_0)) \leq (\phi_0, u_0) \Rightarrow (\phi_0, V(n, x(n))) \leq (\phi_0, u(n)).$$

Theorem 4.1. Assume that there exist functions $V(n, x) \in C[N^+ \times S_\rho, K]$, and $G \in [N^+ \times K, \Re^m]$ such that $V(n, 0) = 0, G(n, 0) = 0, G$ is quasimonotone increasing in u relative to K satisfying the conditions

$H_7)$ $\quad (\phi_0, V(T+1, x(T+1))) \leq (\phi_0, G(n, V(n, x(n))))$, for some $\phi_0 \in K_0^*$.

$H_8)$ $\quad b[(\phi_0, x(n))] \leq (\phi_0, V(n, x(n)) \leq a[(\phi_0, x(n))], \quad a, b \in \mathcal{K},$

Thus

1) If the zero solution of (1.2) is uniformly eventually ϕ_0 -stable, so is the zero solution of (1.1).

2) If the zero solution of (1.2) is uniformly eventually asymptotically ϕ_0-stable, so is the zero solution of (1.1).

Proof. *(1)* Since the zero solution of (1.2)is uniformly eventually ϕ_0 -stable, for given $b(\epsilon) > 0, 0 < \epsilon < \rho, n_0 \in N^+$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$(\phi_0, u_0) < \delta \Rightarrow (\phi_0, u(n)) < b(\epsilon) \quad \text{for} \quad n \geq n_0 \geq \tilde{n}, \tilde{n} > 0 \tag{4.2}$$

By choosing $a[(\phi_0, x_0)] = (\phi_0, u_0)$, we have

$$(\phi_0, V(n, x(n)) \leq a[(\phi_0, x_0)] = (\phi_0, u_0),$$

thus from the condition (H_7) and using Lemma 4.1,we get

$$(\phi_0, V(n, x(n)) \leq (\phi_0, u_0).$$

For $\delta > 0$ such that $a(\delta) = \delta_1$,we have

$$(\phi_0, x_0) < \delta, \quad \text{and} \quad (\phi_0, u_0) = a[(\phi_0, x_0)] < a(\delta) = \delta_1$$

are hold.Thus from(4.2) and the condition (H_8),

$$(\phi_0, x(n))) \leq (\phi_0, V(n, x(n))) \leq (\phi_0, u(n)) < b(\epsilon), \quad \text{for} \quad b \in \mathcal{K},$$

provided that $(\phi_0, u_0) < \delta$.That is

$$(\phi_0, u_0) < \delta \Rightarrow (\phi_0, x(n))) < \epsilon \quad \text{for} \quad n \geq n_0 \geq \tilde{n}, \tilde{n} > 0.$$

Then the zero sulion of (1.1) is uniformly eventually ϕ_0 -stable.

(2) Since the zero solution of (1.2) is uniformly eventually asymptotically ϕ_0 -stable,the zero solution of (1.2) is uniformly eventually ϕ_0 -stable and for given $\epsilon > 0$ there exist $\sigma_1 > 0, T(\epsilon) > 0$ such that

$$(\phi_0, u_0) < \sigma_1 \Rightarrow (\phi_0, u(n))) \leq b(\epsilon) \quad \text{for all} \quad n \geq n_0 + T(\epsilon).$$

Then, in part (1) we can find a positive number $\sigma > 0$ satisfying

$$(\phi_0, x_0) < \sigma, \quad \text{and} \quad (\phi_0, u_0) = a[(\phi_0, x_0)]$$

are hold,it follows that

$$(\phi_0, u_0) < \sigma \Rightarrow (\phi_0, n(n))) < \epsilon \quad \text{for all} \quad n \geq n_0 + T(\epsilon). \tag{4.3}$$

Suppose that this is not true,there exists a divergent sequence $n_j, n_j \geq n_0 + T$,and a solution $x(n, n_0, x_0)$ of the system (1.1) such that for $(\phi_0, x_0) = \epsilon$.From Lemma 4.1,we get the following contradiction

$$b(\epsilon) \leq (\phi_0, V(n, x(n))) \leq (\phi_0, u(n)) < b(\epsilon).$$

Thus(4.3)holds.Therefore the zero solution of (1.1) is uniformly eventually asymptotically ϕ_0 -stable.

Theorem 4.2.Suppose that the conditions of Theorem 4.1 satisfy unless the condition (H_8) is replaced by

$$H_9) \quad b(\| x(n) \|) \leq (\phi_0, V(n, x(n)) \leq a(\| x(n) \|), \quad a, b \in \mathcal{K},$$

Thus

(3) If the zero solution of (1.2) is uniformly eventually ϕ_0- stable, the zero solution of (1.1) is uniformly eventually stable.

(4) If the zero solution of (1.2) is uniformly eventually asymptotically ϕ_0 -stable, the zero solution of (1.1) is uniformly eventually asymptotically stable.

Proof. *(3)* Since the zero solution of (1.2) is uniformly eventually ϕ_0 -stable,for any $\epsilon > 0$,there exits $\tilde{n}, \delta_1 > 0$ independent of n_0 such that

$$(\phi_0, u_0) < \delta \Rightarrow (\phi_0, u(n)) < b(\epsilon) \quad \text{for} \quad n \geq n_0 + T(\epsilon).$$

By choosing $a(\parallel \psi \parallel)$,we have from the condition (H_9)

$$(\phi_0, V(n_0, x_0) \leq a(\parallel \psi \parallel) = (\phi_0, u_0),$$

thus

$$(\phi_0, V(n_0, x_0)) \leq (\phi_0, u_0),$$

From Lemma 4.1,we get

$$(\phi_0, V(n, x(n)) \leq (\phi_0, u_0).$$

For $\delta > 0$ such that $a(\delta) = \delta_1$,we have

$$\parallel x_0 \parallel < \delta, \quad \text{and} \quad (\phi_0, u_0) = a[(\phi_0, u_0)] < \delta_1,$$

thus we get the following contradiction

$$b(\parallel x(n) \parallel) \leq (\phi_0, V(n, x(n))) \leq (\phi_0, u(n)) < b(\epsilon), \quad \text{for} \quad b \in \mathcal{K},$$

therefore

$$\parallel x_0 \parallel < \delta \Rightarrow \parallel x(n) \parallel < \epsilon \quad \text{for} \quad n \geq n_0 \geq \tilde{n}, \tilde{n} > 0.$$

Then the zero sulion of (1.1) is uniformly eventually stable.

(4) Since the zero solution of (1.2) is uniformly eventually asymptotically ϕ_0 -stable,the zero solution of (1.2) is uniformly eventually ϕ_0 -stable and for given $\epsilon > 0$ there exists $\sigma_1 > 0, T(\epsilon) > 0$ that are independent of n_0 such that

$$(\phi_0, u_0) < \sigma_1 \Rightarrow (\phi_0, u(n))) \leq b(\epsilon) \quad \text{for all} \quad n \geq n_0 + T(\epsilon).$$

Then, as in the part (3),for all $n \geq n_0 + T(\epsilon)$, we can find a positive number $\sigma > 0$ satisfying $a(\sigma) < \sigma_1$. It follows that

$$\parallel x_0 \parallel < \sigma \Rightarrow \parallel x(n) \parallel \leq \epsilon \quad \text{for all} \quad n \geq n_0 + T(\epsilon). \tag{4.4}$$

Suppose that this is not true,there exists a divergent sequence $n_j, n_j \geq n_0 + T$,and a solution $x(n, n_0, x_0)$ of the system (1.1) such that

$$\parallel x_0 \parallel < \sigma \Rightarrow \parallel x(n) \parallel = \epsilon.$$

From Lemma 4.1,we get the following contradiction

$$b(\epsilon) \leq (\phi_0, V(n, x(n))) \leq (\phi_0, u(n)) < b(\epsilon).$$

Thus(4.4)holds,therefore the zero solution of (1.1)is uniformly eventually asymptotically stable.

5 Example

Consider the following system[10]

$$\left.\begin{aligned} x_1(n+1) &= a_1x_1 + a_2x_2 + a_1^*x_1^2e^{x_1} + a_2^*x_2^2e^{x_1} + a_1^*a_1^*a_2^*.x_1.x_2.e^{-(x_1+x_2)}. \\ x_2(n+1) &= b_1x_1 + b_2x_2 + b_1^*x_1^2e^{x_2} + b_1^*x_1^2e^{x_2} + b_1^*b_2^*.x_1.x_2e^{-(x_1+x_2)}. \end{aligned}\right\} \tag{5.1}$$

we choose

$$V(n,x(n)) = (V_1, V_2)^T,$$

where $V_1 =| x_1 |$ and $V_2 =| x_2 |$,for $a(s) = s, b(s) = s^2$,we have

$$a(\| x \|) \leq \sum_{j=1}^{j=2} V_j(n,x) \leq b(\| x \|).$$

Thus we can choose

$$G_1(n, V_1, V_2) = a_{11} | x_1 | + a_{12} | x_2 |,$$

and

$$G_2(n, V_1, V_2) = a_{21} | x_1 | + a_{22} | x_2 |$$

it follows that

$$\left.\begin{array}{l} \triangle V_1 \leq G_1(n, V_1, V_2), \\ \triangle V_2 \leq G_2(n, V_1, V_2). \end{array}\right\} \tag{5.2}$$

Therefore we choose

$$\left.\begin{array}{l} u_1(n+1) = a_{11} | u_1 | + a_{12} | u_2 |, \\ u_2(n+1) = a_{21} | u_1 | + a_{22} | u_2 | . \end{array}\right\} \tag{5.3}$$

Supposing that a_{12} and a_{21} are nonnegative.So they are not true. Thus $G(n, u)$ contradicts the quasi-monotone nondecreasing condition in $u = (u_1, u_2)$ relative to $\Re^+$.
Let there exist two numbers η, ξ such that

$$\left.\begin{array}{l} \eta^2 a_{21} + \eta a_{22} \geq \eta a_{11} + a_{12}. \\ \xi^2 a_{21} + \xi a_{22} \geq \xi a_{11} + a_{12}. \end{array}\right\} \tag{5.4}$$

By choosing the cone $K \in \Re^2_+$ defined by

$$K = \{u \in \Re^2_+ \quad \xi u_2 \leq u_1 \leq \eta u_2\}.$$

The boundaries of this cone are $u_1 = \eta u_2$, and $u_2 = \xi u_2$.On the first boundary $u_1 = \eta u_2$,we take

$$\phi_0 = (-\frac{1}{\eta}, 1), \quad \text{So} \quad (-\frac{1}{\eta}, 1)(u_1, \frac{u_1}{\eta}) = 0$$

and

$$(-\frac{1}{\eta}, 1)(a_{11}u_1 + a_{12}\frac{u_1}{\eta}.a_{21}u_1) \geq 0, \quad \text{for} \quad u \neq 0.$$

This reduces to the first condition of(5.4).Similarly,we can get the same result for the second condition of(5.4).Thus the comparison system(5.3) is quasi-monotone relative to the cone K,Then using Theorem 4.2,eventual ϕ_0-stability of the system(5.3)implies eventual stability of the system(5.1).

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